# Ordered Objectives in MaxSAT

*...or when Core-Based MaxSAT Solvers trivialize*

Jeremias Berg, Andre Schidler and Matti Järvisalo

Pragmatics of SAT 2025
11.8.2025

# Motivation

- **MaxSAT** effective optimization paradigm
- Many different optimization algorithms.
  - Focus here: core-based algorithms
- Low-level language $\rightarrow$ intricate encodings

# Motivation

- **MaxSAT** effective optimization paradigm
- Many different optimization algorithms.
  - Focus here: core-based algorithms
- Low-level language $\rightarrow$ intricate encodings

min $z$

**subject to:**

$z \in \{0, 1, \ldots, n\}$

# Motivation

- **MaxSAT** effective optimization paradigm
- Many different optimization algorithms.
  - Focus here: core-based algorithms
- Low-level language → intricate encodings

$$\min \quad z$$

**subject to:**

$$z \in \{0, 1, \ldots, n\}$$

$$\begin{aligned} &\min \quad \textstyle\sum_{k=1}^{n} b_k \\ &\textbf{s.t: } \textstyle\bigwedge_{k=1}^{n} \text{AsCNF}((z \geq k) \leftrightarrow b_k) \end{aligned}$$

$$\begin{aligned} &\min \quad \textstyle\sum_{k=1}^{n} k \cdot b_k \\ &\textbf{s.t: } \textstyle\bigwedge_{k=1}^{n} \text{AsCNF}((z = k) \leftrightarrow b_k) \end{aligned}$$

$$\begin{aligned} &\min \quad \textstyle\sum_{k=1}^{n} b_k \\ &\textbf{s.t: } \textstyle\bigwedge_{k=1}^{n} \text{AsCNF}((z \geq k) \rightarrow b_k) \end{aligned}$$

# Motivating Example

## Cherrypicked instances from MSE 2024

| | EvalMaxSAT-SCIP | EvalMaxSAT-No-SCIP | MaxCDCL | Pacose |
|---|---|---|---|---|
| # solved (UW) | **412** | 404 | 411 | 352 |

# Motivating Example

## Cherrypicked instances from MSE 2024

| | EvalMaxSAT-SCIP | EvalMaxSAT-No-SCIP | MaxCDCL | Pacose |
|---|---|---|---|---|
| # solved (UW) | **412** | 404 | 411 | 352 |

| | EvalMaxSAT-SCIP | EvalMaxSAT-No-SCIP | MaxCDCL | Pacose |
|---|---|---|---|---|
| TWComp_1dj7_N73.wcnf | 1873.42 | 1979.39 | time-out | **992.786** |
| TWComp_1en2_N69.wcnf | 694.287 | 737.122 | 192.768 | **122.647** |
| TWComp_alarm_N37.wcnf | 407.608 | 407.94 | 0.8584 | **0.1516** |
| TWComp_barle_N48.wcnf | 422.69 | 423.961 | 9.9443 | **5.7515** |
| TWComp_barley2_N48.wcnf | 422.692 | 424.69 | 8.7948 | **5.5337** |
| TWComp_celar02_N100.wcnf | 21.4879 | 61.0337 | 62.3124 | **5.8777** |
| TWComp_david_N87.wcnf | 49.522 | 73.1067 | 59.1008 | **39.7042** |
| TWComp_david-pp_N29.wcnf | 407.066 | 409.255 | 2.8107 | **1.0676** |
| TWComp_hailfinder_N56.wcnf | 415.449 | 418.585 | 7.7852 | **0.681** |
| TWComp_mulsol.i.5-pp_N119.wcnf | 271.462 | 356.17 | 110.219 | **58.4436** |
| TWComp_myciel4_N23.wcnf | 445.474 | 446.461 | 18.7783 | **4.5929** |
| TWComp_oesoca4_N42.wcnf | 407.397 | 408.166 | 2.4268 | **0.2316** |
| TWComp_queen5_5_N25.wcnf | 3449.75 | 3277.26 | time-out | **2615.84** |
| TWComp_ship-ship-pp_N30.wcnf | 608.28 | 611.835 | 96.8401 | **28.6641** |
| TWComp_water_N32.wcnf | 407.288 | 412.129 | **1.2079** | 1.4856 |

# Our Contributions

- **(Almost-)Ordered objectives**
  - Relation to MUSes and optimal solutions
  - Large prevalence in MSE and literature.
  - Core-based search trivializes in theory
  - ... and in practice

# Our Contributions

- **(Almost-)Ordered objectives**
  - Relation to MUSes and optimal solutions
  - Large prevalence in MSE and literature.
  - Core-based search trivializes in theory
  - ... and in practice

# Our Contributions

- **(Almost-)Ordered objectives**
  - Relation to MUSes and optimal solutions
  - Large prevalence in MSE and literature.
  - Core-based search trivializes in theory
  - ... and in practice

# Our Contributions

- **(Almost-)Ordered objectives**
  - Relation to MUSes and optimal solutions
  - Large prevalence in MSE and literature.
  - Core-based search trivializes in theory
  - ... and in practice

# Our Contributions

- **(Almost-)Ordered objectives**
  - Relation to MUSes and optimal solutions
  - Large prevalence in MSE and literature.
  - Core-based search trivializes in theory
  - ... and in practice

| | Simple-US | EvalNoSCIP | MaxHS | MaxCDCL | Pacose |
|---|---|---|---|---|---|
| Treewidth (287) | **136** | 126 | 133 | 115 | 85 |
| Judgment Aggregation (50) | **27** | 19 | 23 | 21 | 13 |
| Graph Coloring (415) | 386 | **387** | **387** | 385 | 380 |
| MSE (weighted) (17) | **14** | 13 | **14** | 12 | 11 |
| MSE (unweighted) (108) | **86** | 83 | 86 | 81 | 70 |

# Vocabulary - MaxSAT

## (Weighted Partial) Maximum Satisfiability

$$\min \quad \sum_i c_i \cdot b_i$$

$b_i$ ← objective variable (binary)

$c_i$ ← constant

**subject to:**

$$F$$

$F$ ← CNF-formula

$c_i \cdot b_i$ can be viewed as a soft clause $(\neg b_i)$ of weight $c_i$

## Unsatisfiable Core (Clause)

- Clause $C$ is a core if:
  - $C$ only contains objective variables
  - $F \models C$, i.e. all solutions of $F$ satisfy $C$
  - $F \land \bigwedge_{\ell \in C} (\neg \ell)$ is unsatisfiable

# Vocabulary - MaxSAT

## (Weighted Partial) Maximum Satisfiability

$$\min \quad \sum_i c_i \cdot b_i \leftarrow \text{objective variable (binary)}$$

$c_i$ ← constant

**subject to:**

$$F \longleftarrow \text{CNF-formula}$$

$c_i \cdot b_i$ can be viewed as a soft clause $(\neg b_i)$ of weight $c_i$

## Unsatisfiable Core (Clause)

- Clause $C$ is a core if:
  - $C$ only contains objective variables
  - $F \models C$, i.e. all solutions of $F$ satisfy $C$
  - $F \wedge \bigwedge_{\ell \in C}(\neg \ell)$ is unsatisfiable

MUS a minimal core

# Motivation: Integer Objectives in MaxSAT

**General Problem:**

min $z$

**subject to:**

CONSTRAINTS
$z \in \{0, 1, \ldots, n\}$

### Example Domains

Graph Coloring, Treewidth, Judgment Aggregation, Clustering, Database Queries, . . .

# Motivation: Integer Objectives in MaxSAT

**General Problem:**

min $z$ ← integer variable

**subject to:**

$\textsc{Constraints}$ ← general constraints

$z \in \{0, 1, \ldots, n\}$

↑ finite domain

## Example Domains

Graph Coloring, Treewidth, Judgment Aggregation, Clustering, Database Queries, ...

# Motivation: Integer Objectives in MaxSAT

**General Problem:**

min $z$

**subject to:**

CONSTRAINTS
$z \in \{0, 1, \ldots, n\}$

**As MaxSAT:**

min $\sum_{k=1}^{n} b_k$ ← binary variable

**subject to:**

$\bigwedge_{k=1}^{n}$ ASCNF($(z \geq k) \leftrightarrow b_k$)
ASCNF(CONSTRAINTS)

↑ CNF formulas

## Example Domains

Graph Coloring, Treewidth, Judgment Aggregation, Clustering, Database Queries, . . .

# Ordered Objectives

## Definition

- Solutions order objective.
  - $b_k \rightarrow b_{k-1}$ for all solutions.

## Properties

- MUSes are units.
- MUSes are ordered.

$$\min \quad \sum_{k=1}^{n} b_k$$

**subject to:**

$$\bigwedge_{k=1}^{n} \textsc{AsCNF}((z \geq k) \leftrightarrow b_k)$$

$$\textsc{AsCNF}(\textsc{Constraints})$$

# Ordered Objectives

## Definition

- Solutions order objective.
  - $b_k \rightarrow b_{k-1}$ for all solutions.

## Properties

- MUSes are units.
- MUSes are ordered.

$$\min \quad \sum_{k=1}^{n} b_k$$

**subject to:**

$$\bigwedge_{k=1}^{n} \textsc{AsCNF}((z \geq k) \leftrightarrow b_k)$$

$$\textsc{AsCNF}(\textsc{Constraints})$$

# Ordered Objectives

## Definition

- Solutions order objective.
  - $b_k \rightarrow b_{k-1}$ for all solutions.

## Properties

- MUSes are units.
  - e.g. $(b_k \lor b_{k-1})$ is not an MUS.
- MUSes are ordered.
  - if $(b_k)$ is an MUS, so is $(b_{k-1})$.

min $\sum_{k=1}^{n} b_k$

**subject to:**

$\bigwedge_{k=1}^{n}$ AsCNF$((z \geq k) \leftrightarrow b_k)$

AsCNF(Constraints)

# Ordered Objectives

## Definition

- Solutions order objective.
  - $b_k \rightarrow b_{k-1}$ for all solutions.

## Properties

- MUSes are units.
  - e.g. $(b_k \vee b_{k-1})$ is not an MUS.
- MUSes are ordered.
  - if $(b_k)$ is an MUS, so is $(b_{k-1})$.

min $\sum_{k=1}^{n} b_k$

**subject to:**

$$\bigwedge_{k=1}^{n} \text{AsCNF}((z \geq k) \leftrightarrow b_k)$$

$$\text{AsCNF}(\text{Constraints})$$

# Motivation: Integer Encodings with Implications

**General Problem:**

min $\quad z$

**subject to:**

CONSTRAINTS
$z \in \{0, 1, \ldots, n\}$

**As MaxSAT (in theory):**

min $\quad \sum_{k=1}^{n} b_k$

**subject to:**

equivalence ↘

$\bigwedge_{k=1}^{n}$ ASCNF($(z \geq k) \leftrightarrow b_k$)
ASCNF(CONSTRAINTS)

# Motivation: Integer Encodings with Implications

**General Problem:**

min $z$

**subject to:**

CONSTRAINTS
$z \in \{0, 1, \ldots, n\}$

**As MaxSAT (in practice):**

min $\sum_{k=1}^{n} b_k$

**subject to:**

implication ↘

$\bigwedge_{k=1}^{n} \text{AsCNF}((z \geq k) \rightarrow b_k)$
$\text{AsCNF}(\text{Constraints})$

# Almost-Ordered Objectives

## Definition

- Optimal solutions order objective.
  - $b_k \to b_{k-1}$ for all opt. sols.

## Properties

- All solutions assign the objective the same way.

$$\min \quad \sum_{k=1}^{n} b_k$$

**subject to:**

$$\bigwedge_{k=1}^{n} \text{AsCNF}((z \geq k) \to b_k)$$

$$\text{AsCNF}(\text{Constraints})$$

# Almost-Ordered Objectives

**Definition**

- Optimal solutions order objective.
  - $b_k \to b_{k-1}$ for all opt. sols.

**Properties**

- All solutions assign the objective the same way.
  - no opt sol assigns $b_2 = 1$ and $b_1 = 0$.

$$\min \quad \sum_{k=1}^{n} b_k$$

**subject to:**

$$\bigwedge_{k=1}^{n} \textsc{AsCNF}((z \geq k) \to b_k)$$

$$\textsc{AsCNF}(\textsc{Constraints})$$

# Summary: (Almost-)Ordered Objectives

## Ordered Objective

$(F, \sum_i c_i b_i)$ has an ordered objective if:

- all solutions of $F$ satisfy $(b_i \rightarrow b_{i-1})$

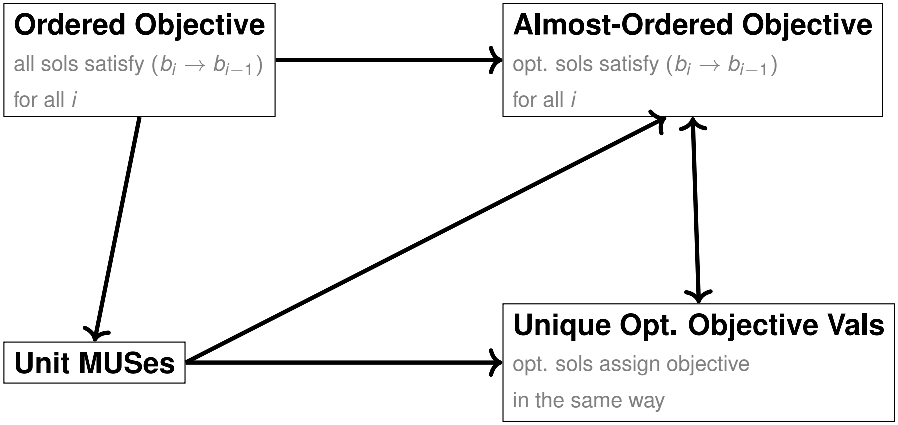
## Almost Ordered Objective

$(F, \sum_i c_i b_i)$ has an almost-ordered objective if:

- every optimal solution satisfies $(b_i \rightarrow b_{i-1})$

# Summary: Properties of (Almost-)Ordered Objectives

Formula $F$, objective $\sum_i c_i b_i$



# Prevalence of Ordered Objectives

## Instances of MSE 22-24

### Algorithmic Detection

- **Ordered:** Exhaustive search or UP-based lookahead.
- **Almost-Ordered:** Enumerating MUSes or optimal solutions.

# Prevalence of Ordered Objectives

## Instances of MSE 22-24

### Algorithmic Detection

- **Ordered:** Exhaustive search or UP-based lookahead.
- **Almost-Ordered:** Enumerating MUSes or optimal solutions.

| Instances | # | Ord. | Not Ord. | AO | Not AO |
|---|---|---|---|---|---|
| Unweighted | 1558 | **134** | 1266 | **589** | 556 |
| Weighted | 1545 | **29** | 1421 | **616** | 439 |

# Prevalence of Ordered Objectives

## Instances of MSE 22-24

### Algorithmic Detection

- **Ordered:** Exhaustive search or UP-based lookahead.
- **Almost-Ordered:** Enumerating MUSes or optimal solutions.

| Instances | # | Ord. | Not Ord. | AO | Not AO |
|---|---|---|---|---|---|
| Unweighted | 1558 | **134** | 1266 | **589** | 556 |
| Weighted | 1545 | **29** | 1421 | **616** | 439 |

Over a third of instances determined to have an almost-ordered objective

# Solving instances with an ordered objective

## Intuition

- optimal solutions set $b_1 = b_2 = \ldots = b_t = 1$ and rest to 0.
- suffices to find $t$.

## Simple-US

```
Simple-US(F, ∑_i c_i b_i)
    for i=1,...,n do
        (sat?, α) ← SAT-Solve-w-Assums(F, {b_i = 0});
        if sat? then return α;
```

# Solving instances with an ordered objective

## Intuition

- optimal solutions set $b_1 = b_2 = \ldots = b_t = 1$ and rest to 0.
- suffices to find $t$.

## `Simple-US`

```
Simple-US(F, ∑_i c_i b_i)
    for i=1,...,n do
        (sat?, α) ← SAT-Solve-w-Assums(F, {b_i = 0});
        if sat? then return α;
```

# Core-Guided MaxSAT

Core-Guided *(F, O)*
 $(F^W, O^W) \leftarrow (F, O)$
 **while** TRUE **do**
  $\mathcal{A} \leftarrow \{\neg b \mid c \cdot b \in O^W\}$
  $(sat?, \alpha, C) \leftarrow \texttt{ExtractCore-w-SAT-solver}(F^W, \mathcal{A})$
  **if** *sat?* **then return** $\alpha$;
  $(F^W, O^W) \leftarrow \texttt{Relax}(F^W, O^W, C)$

## Intuition for this talk

- Least number of iterations when extracting MUSes.
- Unit cores fixed.

# State-of-the-art Core-Based MaxSAT

## On ordered objectives

### Informal Theorem

`Simple-US` simulates best-case of state-of-the-art core-guided and IHS MaxSAT solvers.

### `Simple-US`

`Simple-US` *(F, O)*
  **for** *i=1,...,n* **do**
    $(sat?, \alpha) \leftarrow$ `SAT-Solve-w-Assums`$(F, \{b_i = 0\})$;
    **if** *sat?* **then return** $\alpha$;
    /* Else $F \leftarrow F \wedge (b_i)$ */

# Back to this Table

| | Simple-US | EvalNoSCIP | MaxHS | MaxCDCL | Pacose |
|---|---|---|---|---|---|
| Treewidth (287) | **136** | 126 | 133 | 115 | 85 |
| Judgment Aggregation (50) | **27** | 19 | 23 | 21 | 13 |
| Graph Coloring (415) | 386 | **387** | **387** | 385 | 380 |
| MSE (weighted) (17) | **14** | 13 | **14** | 12 | 11 |
| MSE (unweighted) (108) | **86** | 83 | 86 | 81 | 70 |

# Conclusions

## (Almost-)Ordered Objectives

- Connections to MUS structure and optimal solutions.
- High prevalence
- Trivialize state-of-the-art core-guided and IHS solvers.

# Conclusions

## (Almost-)Ordered Objectives

- Connections to MUS structure and optimal solutions.
- High prevalence
- Trivialize state-of-the-art core-guided and IHS solvers.

**Take Home Message:**
If your encoding results in an ordered objective, core-based solvers are (probably) not the best idea.

# Details on Runtime of Empirical Solvers

| Solver | UW (108) # | UW (108) PAR-2 | W (17) # | W (17) PAR-2 | Col (415) # | Col (415) PAR-2 | JA AO (50) # | JA AO (50) PAR-2 | JA O (50) # | JA O (50) PAR-2 | TW AO (287) # | TW AO (287) PAR-2 | TW O (287) # | TW O (287) PAR-2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| `Simple-SIS` | 88 | 1780.3 | 13 | 2360.7 | 387 | 497.1 | 27 | 3819.5 | 27 | 3826.1 | 142 | 3818.9 | 143 | 3807.5 |
| `Simple-US` | 86 | 1950.0 | 14 | 1760.1 | 386 | 510.6 | 20 | 4827.9 | 27 | 4067.0 | 142 | 3864.1 | 136 | 3990.6 |
| EvalSCIP | 84 | 2248.1 | 12 | 2468.4 | 392 | 481.9 | 18 | 5160.3 | 19 | 5031.4 | 121 | 4441.8 | 123 | 4415.3 |
| EvalNoSCIP | 83 | 2135.0 | 13 | 2271.6 | 387 | 498.8 | 20 | 4940.8 | 19 | 4962.6 | 124 | 4228.2 | 126 | 4183.9 |
| MaxHS | 86 | 1959.1 | 14 | 1985.2 | 387 | 501.6 | 12 | 5777.3 | 23 | 4442.7 | 125 | 4233.0 | 133 | 4027.5 |
| MaxCDCL | 81 | 2360.4 | 12 | 2628.6 | 385 | 606.6 | 21 | 5099.3 | 21 | 4949.1 | 108 | 4660.4 | 115 | 4502.4 |
| Pacose | 70 | 2876.5 | 11 | 3304.3 | 380 | 637.5 | 12 | 5862.8 | 13 | 5663.0 | 77 | 5417.4 | 85 | 5167.0 |
| Virtual Best | 89 | 1720.4 | 14 | 1603.5 | 393 | 388.3 | 28 | 3733.2 | 28 | 3692.8 | 143 | 3793.4 | 143 | 3792.6 |

# Overhead on Clean Reimplementations

| | Col (415) | | JA AO (50) | | JA O (50) | | TW AO (287) | | TW O (287) | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Non-unit | Overh. | Non-unit | Overh | Non-unit | Overh | Non-unit | Overh | Non-unit | Overh |
| IHS | 140/383 | 1.2 | 5/5 | 9.2 | 6/6 | 2.6 | 43/65 | 1.2 | 41/66 | 1.2 |
| OLL | 0/387 | 0 | 21/21 | 41 | 27/27 | 6 | 66/125 | 10 | 59/135 | 1 |

# Details on Prelevance of Almost-Ordered Objectives

| | | Ordered | | Not ordered | Almost-ordered | | Not almost-ordered |
|---|---|---|---|---|---|---|---|
| Instances | # | UP | Total | | UMUS | Unique | |
| Unweighted | 1558 | 113 | 134 | 1266 | 267 | 589 | 556 |
| Weighted | 1545 | 17 | 29 | 1421 | 93 | 616 | 439 |