Quantum Graph-State Synthesis with SAT

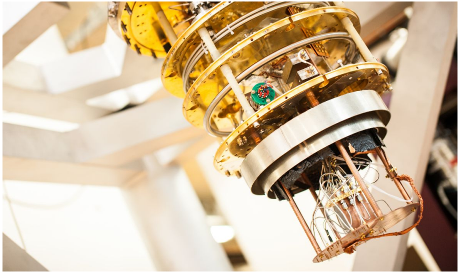
Sebastiaan Brand   Tim Coopmans   Alfons Laarman

Pragmatics of SAT, 4 July 2023

Universiteit Leiden
The Netherlands

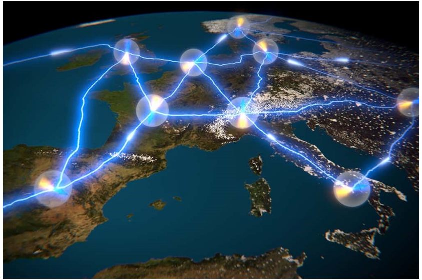
Quantum computing & networking

Quantum computing



- polynomial speedup to a lot of problems (including SAT)
- exponential speedup to some problems

Quantum networking



- better cryptography
- connect quantum computers

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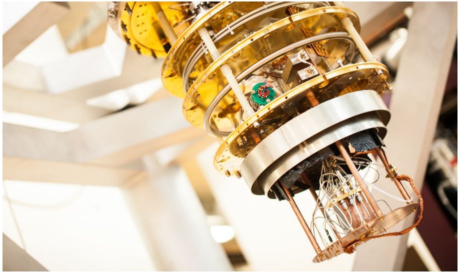

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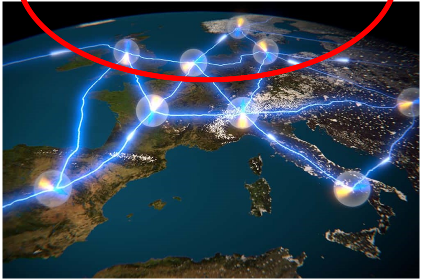

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Graph states

Quantum states represented by (undirected) graphs

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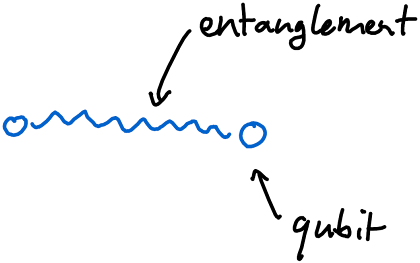

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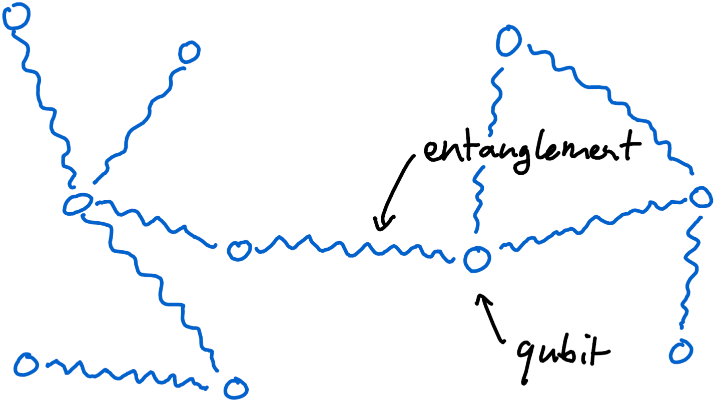
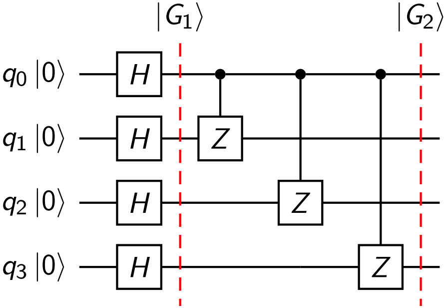


Graph states



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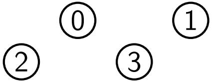
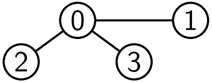


(b) Graph G_1.



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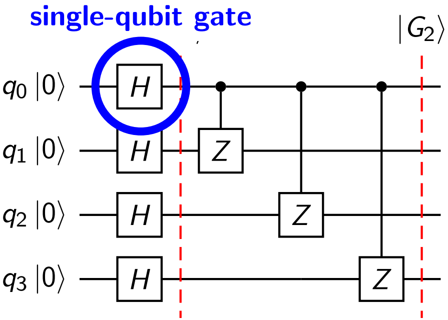
Graph states



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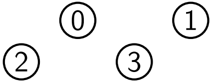

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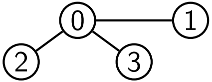

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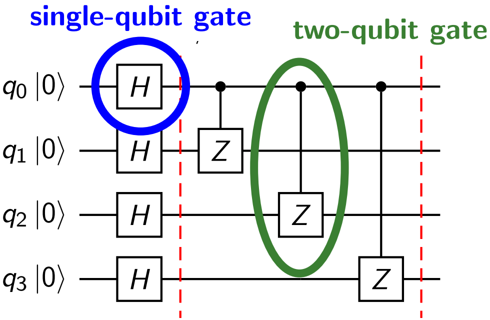

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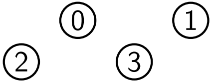

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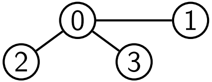

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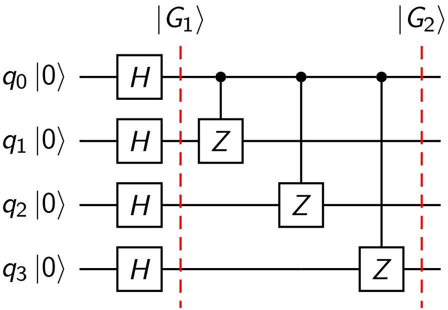

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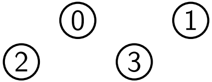

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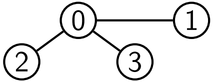
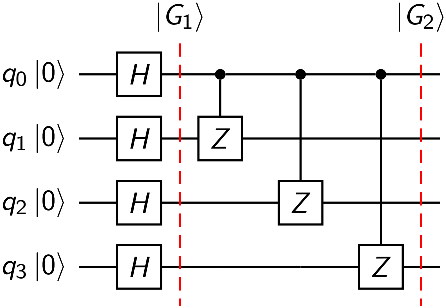


(c) Graph G_2.

$$|G_1\rangle = \frac{1}{4}(|0000\rangle + |0001\rangle + \cdots + |1011\rangle + |1100\rangle + |1101\rangle + |1110\rangle + |1111\rangle)$$

Graph states



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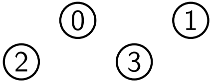

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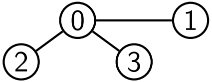

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Graph-state synthesis - example problem

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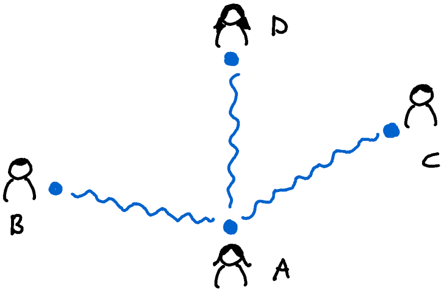
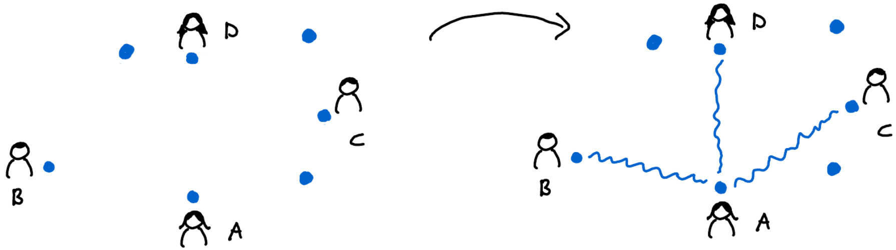


They need to share a specific quantum state (the GHZ state), described by this graph

$$|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

Graph-state synthesis - problem definition



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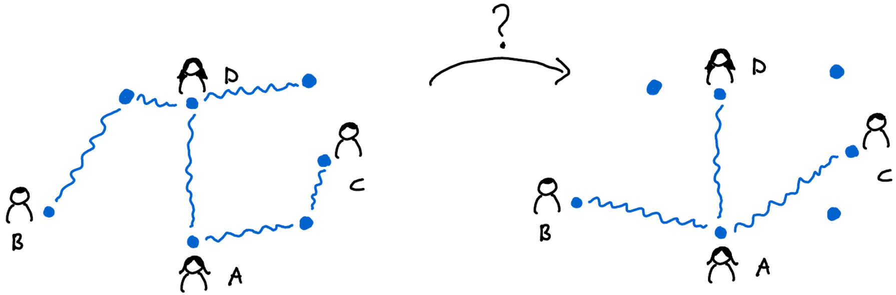
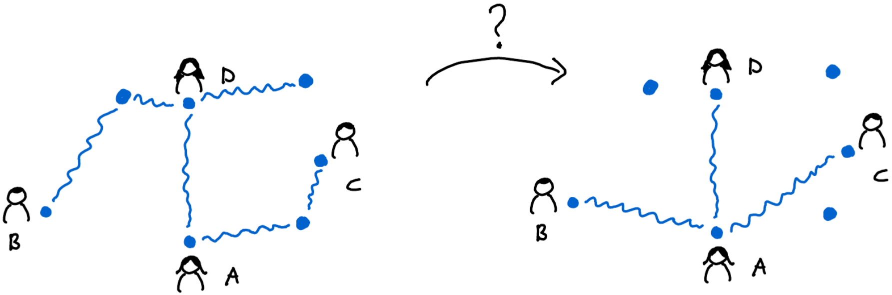


Graph-state synthesis - problem definition

Goal: generate a target graph, given some initial entanglement, **using only local operations**



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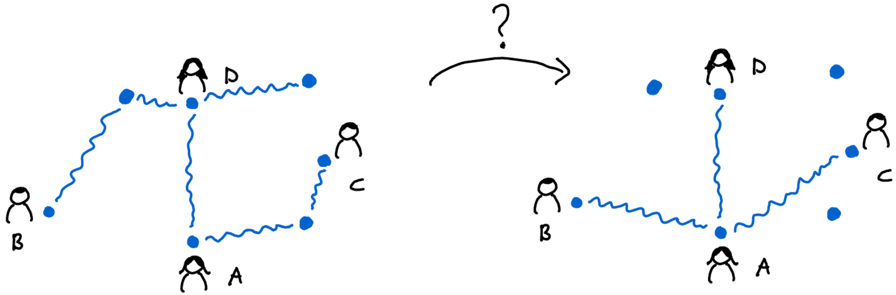

This problem has been shown to be NP-complete[1]

[1] Dahlberg, A. et al. (2020). How to transform graph states using single-qubit operations: computational complexity and algorithms.

Graph-state synthesis - local operations

local quantum operations

corresponding graph operations

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single-qubit quantum gates

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local quantum operations | **corresponding graph operations**

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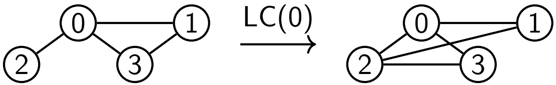
Graph-state synthesis - local operations

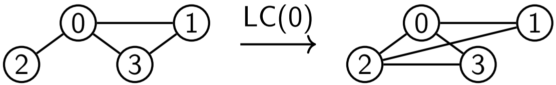
local quantum operations	**corresponding graph operations**
single-qubit quantum gates	**local complementations**
single-qubit measurements	



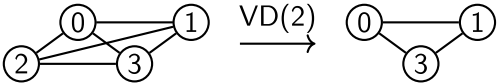
Graph-state synthesis - local operations

local quantum operations	**corresponding graph operations**
single-qubit quantum gates	**local complementations**
single-qubit measurements	**vertex deletions**

local complementations



vertex deletions



SAT encoding

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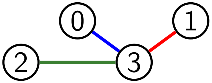
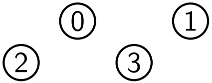
Examples:



encoded as $\neg x_{01} \wedge \neg x_{02} \wedge \mathbf{x_{03}} \wedge \neg x_{12} \wedge \mathbf{x_{13}} \wedge \mathbf{x_{23}}$



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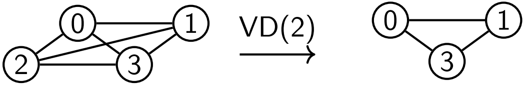
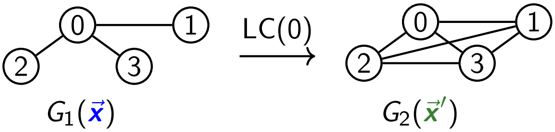


$G_1(\vec{x})$ $\quad$ $G_2(\vec{x}')$

$$VD_k = \bigwedge_{(u,v)\in\mathbb{U}} \begin{cases} \neg \mathbf{x}'_{\mathbf{uv}} & \text{if } u = k \text{ or } v = k \\ \mathbf{x}'_{\mathbf{uv}} \leftrightarrow \mathbf{x}_{\mathbf{uv}} & \text{otherwise.} \end{cases}$$

SAT encoding

local complementation



$$LC_k = \bigwedge_{(u,v)\in\mathbb{U}} \begin{cases} \mathbf{x}'_{\mathbf{uv}} \leftrightarrow \neg((\mathbf{x}_{\mathbf{uk}} \wedge \mathbf{x}_{\mathbf{vk}}) \oplus \neg \mathbf{x}_{\mathbf{uv}}) & \text{if } u \neq k \text{ and } v \neq k \\ \mathbf{x}'_{\mathbf{uv}} \leftrightarrow \mathbf{x}_{\mathbf{uv}} & \text{otherwise.} \end{cases}$$

SAT encoding

$$\left.\begin{array}{l} LC_0, LC_1, \ldots, LC_{n-1} \\ VD_0, VD_1, \ldots, VD_{n-1} \end{array}\right\}$$ combine into single global transition relation $R(\vec{x}, \vec{x}')$

$R(\vec{x}, \vec{x}')$ encodes all 1-step transformations and has $n^2 + \log n$ variables and $3.5n^3 + O(n^2 \log n)$ clauses.

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To encode multiple sequential transformation steps, we can use bounded model checking:

$$\overbrace{S(\vec{x}_1)}^{\text{starting graph}} \quad \wedge \quad \underbrace{R(\vec{x}_1, \vec{x}_2) \wedge R(\vec{x}_2, \vec{x}_3) \wedge \cdots \wedge R(\vec{x}_{d-1}, \vec{x}_d)}_{\text{any sequence of LC+VD of length } d} \quad \wedge \quad \overbrace{T(\vec{x}_d)}^{\text{target graph}}$$

Completeness threshold

Theorem

If a transformation from G_s to G_t exists using local complementations and vertex deletions, a transformation of length $\leq 2.5n$ exists, where n is the number of vertices in G_s.

Proof (sketch)

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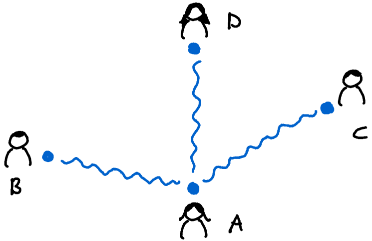
Proof (sketch)

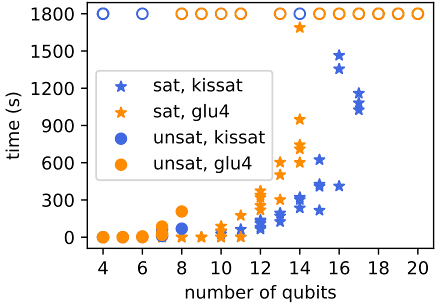
1. *All local complementations can be done before the vertex deletions*
2. *We know exactly which vertex deletions need to happen*
3. *We can bound the number of local complementations*[2]

[2] Bouchet, A. (1991). An efficient algorithm to recognize locally equivalent graphs.

Results

Synthesize the 4-qubit GHZ state from a random graph of n qubits





For comparison, various graph-state properties have been explored numerically up to 12 qubits[3]

[3] Cabello, A. et al. (2011). Optimal preparation of graph states.

Adding non-local operations

non-local quantum operations

corresponding graph operations

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two-
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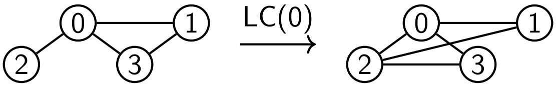
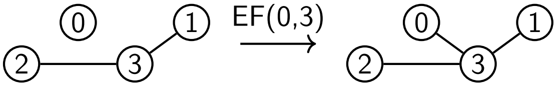


+

edge flips



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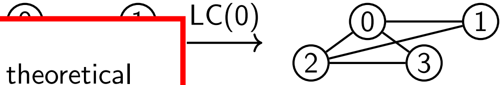

significantly fewer theoretical results in this regime

+

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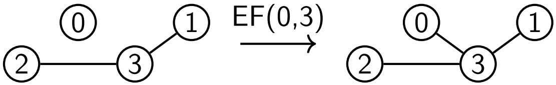
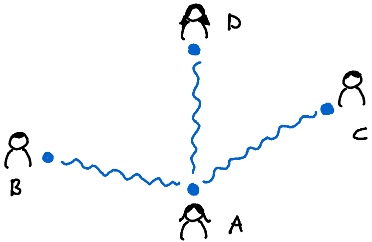
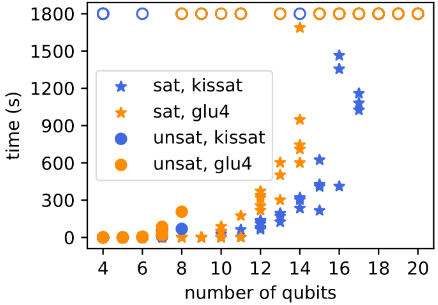


Adding non-local operations

Synthesize the 4-qubit GHZ state from a random graph of n qubits, **allowing for two-qubit operations** on a limited subset of $V \times V$.





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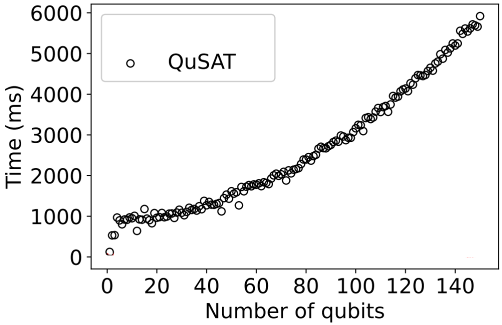
E.g. Clifford circuit equivalence checking:



[4] Berent, L., Burgholzer, L., Wille, R. (2022). Towards a SAT encoding for quantum circuits: A journey from classical circuits to Clifford circuits and beyond. arXiv preprint arXiv:2203.00698.

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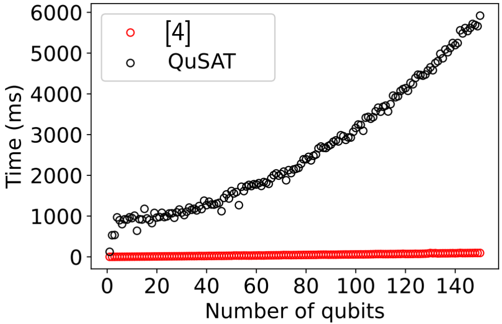

[4] Thanos, D., Coopmans, T., Laarman, A. (2023) Fast equivalence checking of quantum circuits of Clifford gates. *To appear at ATVA 2023*

Summary

1. **Graph states** are an important subset of quantum states with many applications, e.g. in quantum networking
2. We want to synthesize graph states using **local operations** because these are easier to do
3. We translate this NP-complete problem to SAT and are able to find graph state transformations up to **17 qubits**
4. The method easily generalizes to **non-local operations**