Certified Symmetry and Dominance Breaking for Combinatorial Optimisation

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and Lund University

13th Pragmatics of SAT workshop
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Joint *AAAI '22* paper with Bart Bogaerts, Stephan Gocht, and Ciaran McCreesh

Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong... (Even best commercial ones) [BLB10, CKSW13, AGJ^{+}18, GSD19, GS19]
- Software testing doesn't suffice to resolve this problem
- Formal verification techniques cannot deal with level of complexity of modern solvers

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Design certifying algorithms [ABM+11, MMNS11] that

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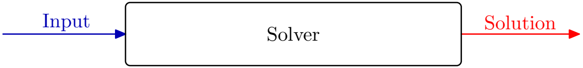

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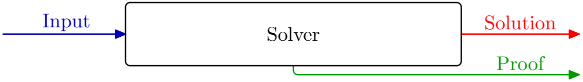

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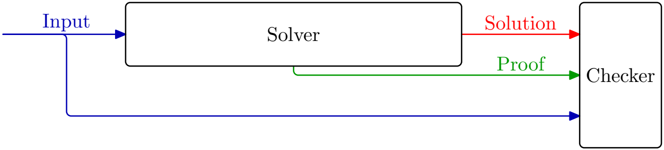

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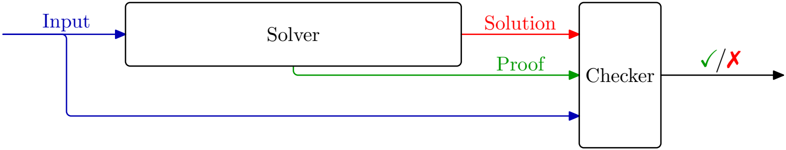

Workflow:

1. Run solver on problem input
2. Get as output not only solution but also proof
3. Feed input + solution + proof to proof checker
4. Verify that proof checker says solution is correct

Yet Another SAT Success Story

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH+17]
- ...

Well established — required in main track of SAT competitions

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But efficient proof logging has remained out of reach for stronger paradigms

And, in fact, even for some advanced SAT solving techniques:

- cardinality reasoning
- Gaussian elimination
- symmetry handling

Clausal Proof Logging Approaches

Cardinality and pseudo-Boolean reasoning [SB06, BBH22]
Evaluated on fairly specific crafted benchmarks
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Symmetry handling [HHW15, TD20]
No fully general method for symmetry breaking (i.e., adding constraints to remove symmetric solutions)
Method for symmetric learning (i.e., adding symmetric versions of derived constraints) not compatible with SAT preprocessing

Our Work: Efficient Proof Logging for Symmetry Breaking

Paper *Certified Symmetry and Dominance Breaking for Combinatorial Optimisation* at *AAAI '22* [BGMN22]:

Implementation in proof checker VeriPB [Ver]

- First general & efficient proof logging method for symmetry breaking
- Supports also pseudo-Boolean reasoning and Gaussian elimination
- Based on 0-1 integer linear constraints instead of clauses
- Uses cutting planes method [CCT87] with additional rules

Outline of Presentation

What I hope to cover in the rest of this presentation:

- Basics of proof logging with 0-1 linear constraints
- New rule for symmetry and dominance breaking
- Application to symmetry breaking for SAT (and some other problems)
- Some future research directions

0-1 Integer Linear (a.k.a. Pseudo-Boolean) Constraints

Pseudo-Boolean (PB) constraints are 0-1 integer linear constraints

$$C \doteq \sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Pseudo-Boolean formulas $F \doteq \bigwedge_{i=1}^{m} C_i$ are conjunctions of pseudo-Boolean constraints

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

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(See [BN21] for more details about cutting planes)

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- Generalize **reverse unit propagation (RUP)** rule [GN03, Van08] to PB constraints — just convenient shorthand for derivation
- Also need **extension** rule (analogue of RAT [JHB12]) to deal with, e.g., preprocessing

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C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

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- Implication should be efficiently verifiable — every $D \in (F \wedge C)\restriction_{\omega}$ should follow from $F \wedge \neg C$ by, e.g.,
  1. weakening (addition of literal axioms $\ell_i \geq 0$)
  2. reverse unit propagation (RUP)
  3. explicit derivation presented in proof log

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Don't miss CP tutorial Tue Aug 2 at 14:00 *Solving with Provably Correct Results: Beyond Satisfiability, and Towards Constraint Programming*

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(Syntactic) symmetry: substitution σ preserving F ($F\restriction_\sigma \doteq F$)

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Deal with symmetry breaking by switching focus to optimisation
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Note that $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ means $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$

Proof Logging for Optimisation Problems

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Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega \;\wedge\; f\restriction_\omega \leq f$$

Redundance and Dominance Rules

Redundance-based strengthening, optimisation version

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \wedge C$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive also C_m if exists witness substitution ω such that

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F{\restriction_\omega} \wedge f{\restriction_\omega} < f$$

Only consider original formula — no need to show that any $C_i{\restriction_\omega}$ implied!

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Now why is **this** sound?

- Same inductive proof as before, but nested
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Further extensions:

- Define dominance rule w.r.t. order independent of objective function
- Switch between different orders in same proof
- See [BGMN22] for details

Strategy for SAT Symmetry Breaking

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)

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3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$y_0 \qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$
$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$
$$\overline{y}_j \vee y_{j-1} \qquad y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$$

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f{\restriction}_\sigma$ can be derived from F using dominance with witness σ

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Redundance-based strengthening can be used analogously to [HHW15]

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- but only guaranteed to work for breaking single symmetry σ
- if σ is involution (i.e., its own inverse)
- not known how to deal with symmetries that are complex or interact

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple

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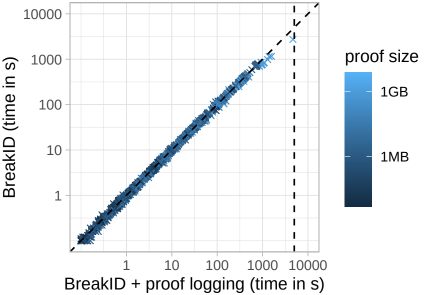
$$F \wedge C_\sigma \wedge \neg C_\tau \models F\restriction_\tau \wedge f\restriction_\tau < f$$

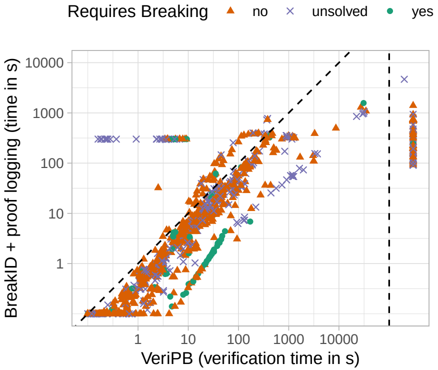
Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

Experimental Evaluation

- Evaluated on SAT competition benchmarks
- BreakID [DBBD16, Bre] used to find and break symmetries





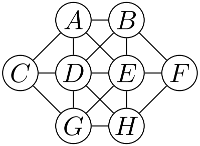
- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

Symmetry Breaking for Constraint Programming

Crystal Maze puzzle

Place numbers 1 to 8 without repetition

Adjacent circles mustn't have consecutive numbers



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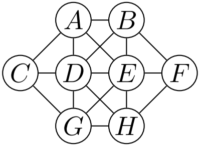

Without loss of generality:

- $A < G$ (horizontal mirror symmetry)
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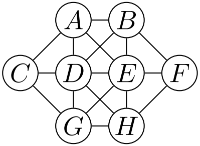

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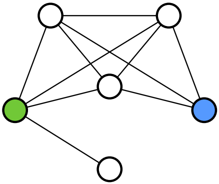
Technical challenge: integer-valued variables
See [GMN22] for more detailed discussion

Dominance Breaking for Maximum Clique Solving

Maximum clique solving
Find largest fully connected component



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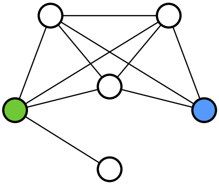

Lazy global domination [MP16]
Only consider green and not blue vertex
(since every neighbour of blue is also neighbour of green)

Technical challenge: vertex domination detected only lazily during search
Dominance rule (rather than redundance rule) really helpful here

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress)*

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- Symmetric learning and recycling (substitution) of subproofs
- Maximum satisfiability (MaxSAT) solving *(work in progress [VWB22])*
- Pseudo-Boolean optimization
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And more. . .

- Lots of challenging problems and interesting ideas
- We're hiring! Talk to me to join the proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **This work:** Efficient proof logging for symmetry and dominance breaking using cutting planes with extensions

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Thank you for your attention!

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