Leveraging Linear Programming for pseudo-Boolean solving

Jo Devriendt †, Jan Elffers †, Ambros Gleixner ‡, Jakob Nordström *
† KTH Royal Institute of Technology, Sweden
‡ Zuse Institut Berlin, Germany
* University of Copenhagen, Denmark
jhmde@kth.se

Pseudo-Boolean background

- Pseudo-Boolean (PB) **constraint**:
  - Bounded weighted sum of literals:

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- Rationally infeasible **implies UNSAT**

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For CNF, deciding rational infeasibility is trivial

Pseudo-Boolean background

- Deciding rational infeasibility of PB formulas is **easy in theory**:
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Goal of our work:

use LP solver to check rational feasibility during PB search

Linear Programming (LP) solver

- Input:
  - conjunction of linear constraints
  - variable bounds
  - objective function

$$
\begin{array}{ll}
\text{minimize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
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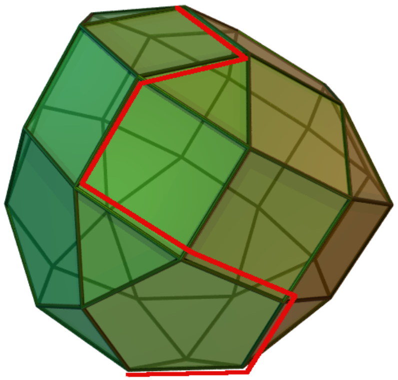

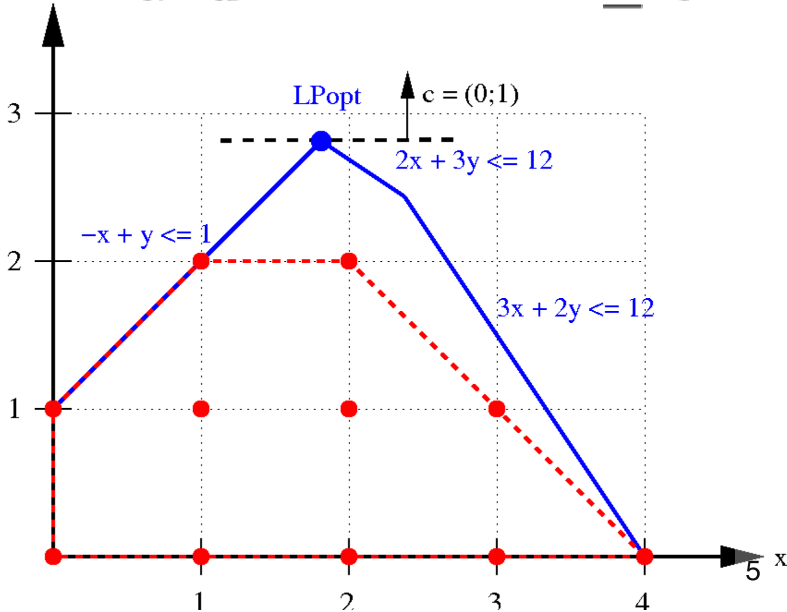

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    - defines violated positive linear combination of input constraints

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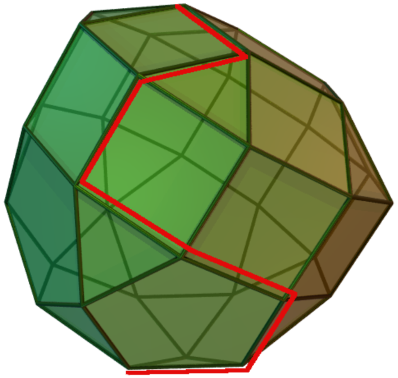

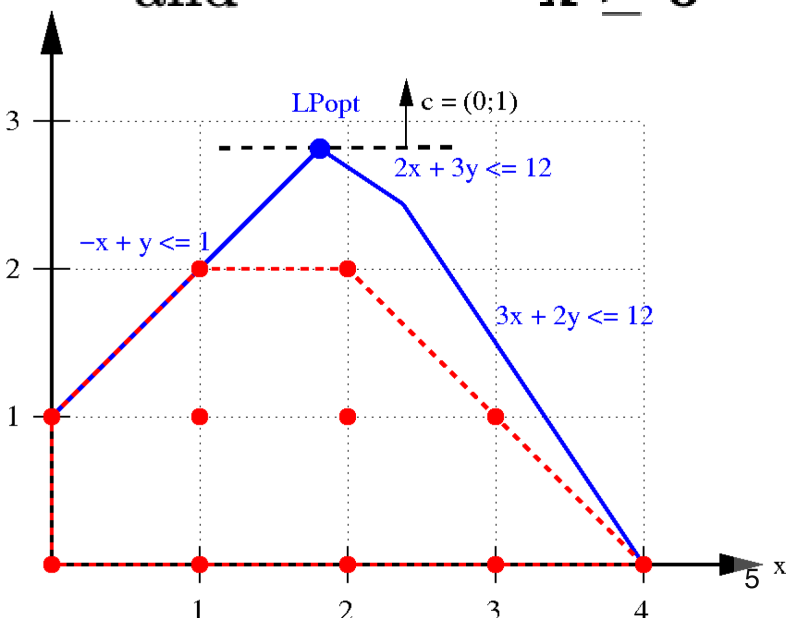

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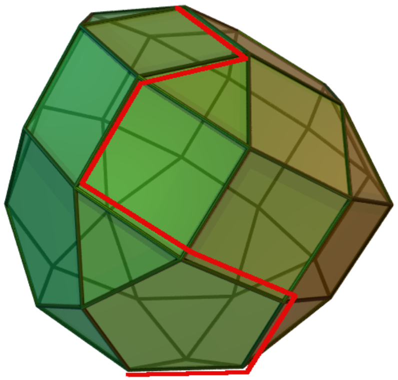
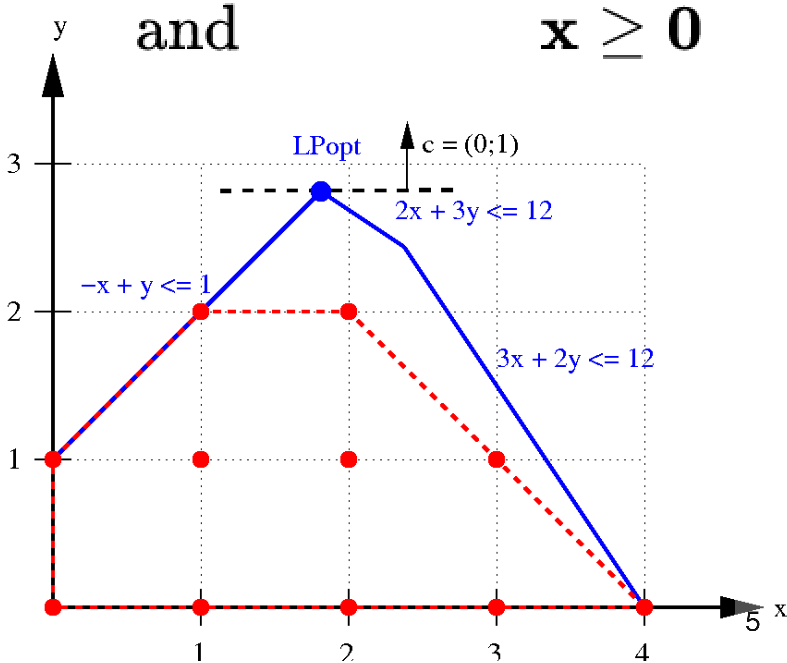




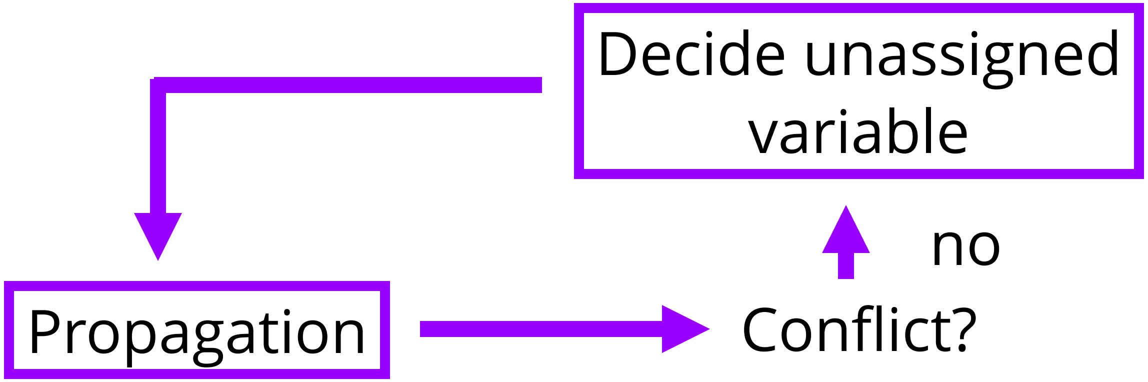
PB search loop

Propagation

PB search loop

Propagation → Conflict?

PB search loop



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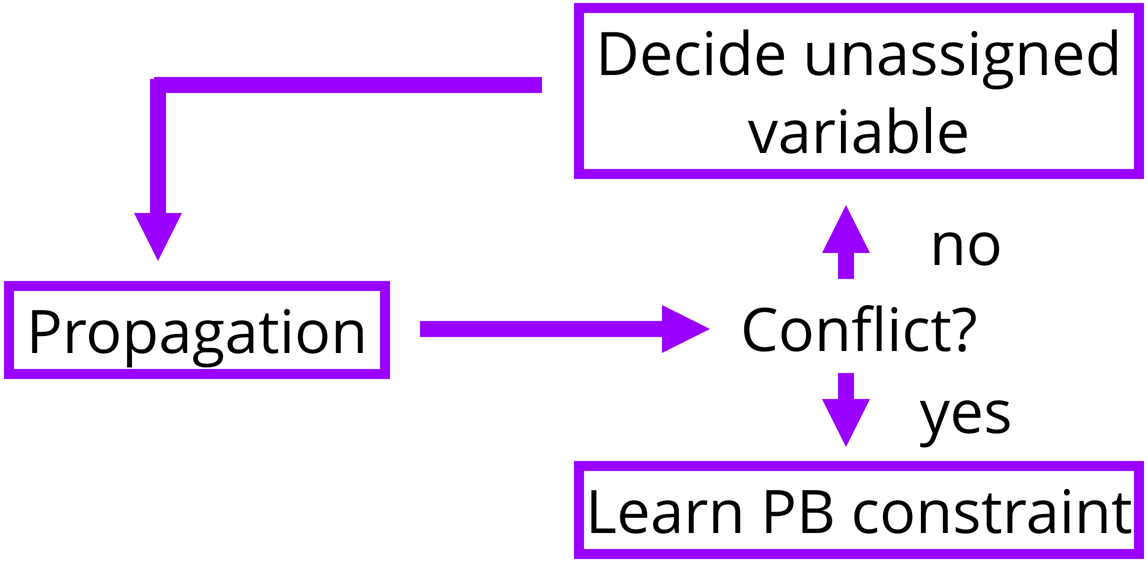

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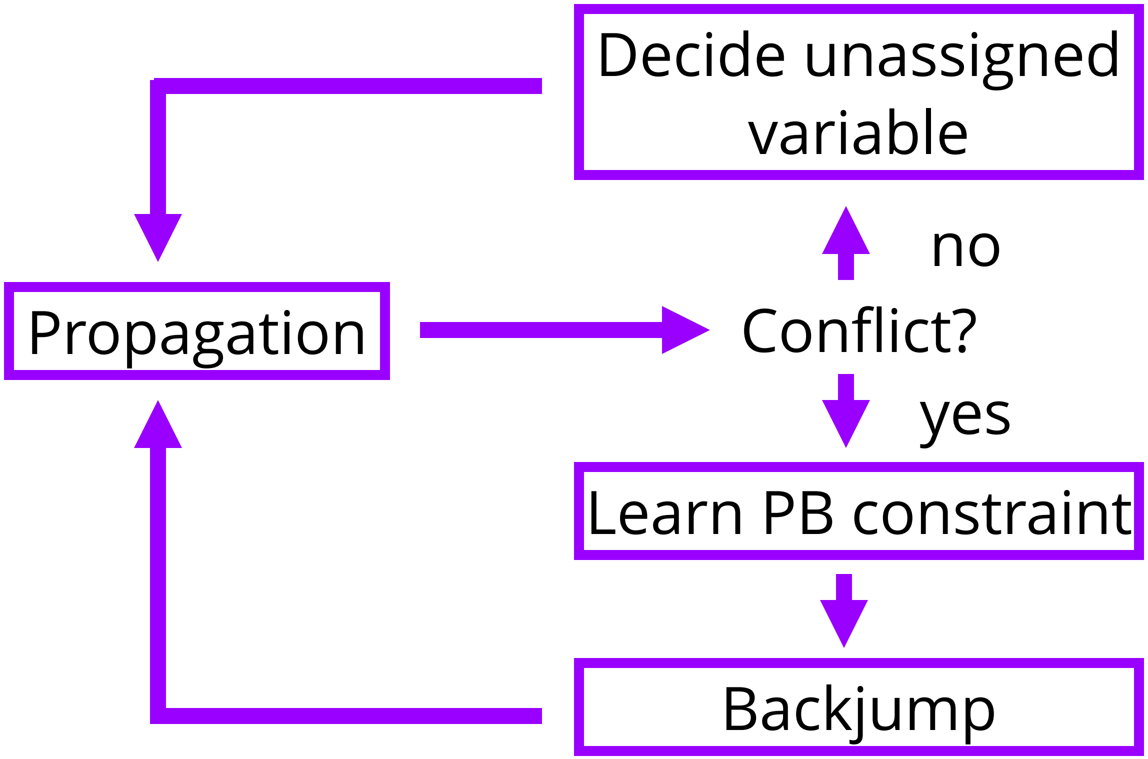
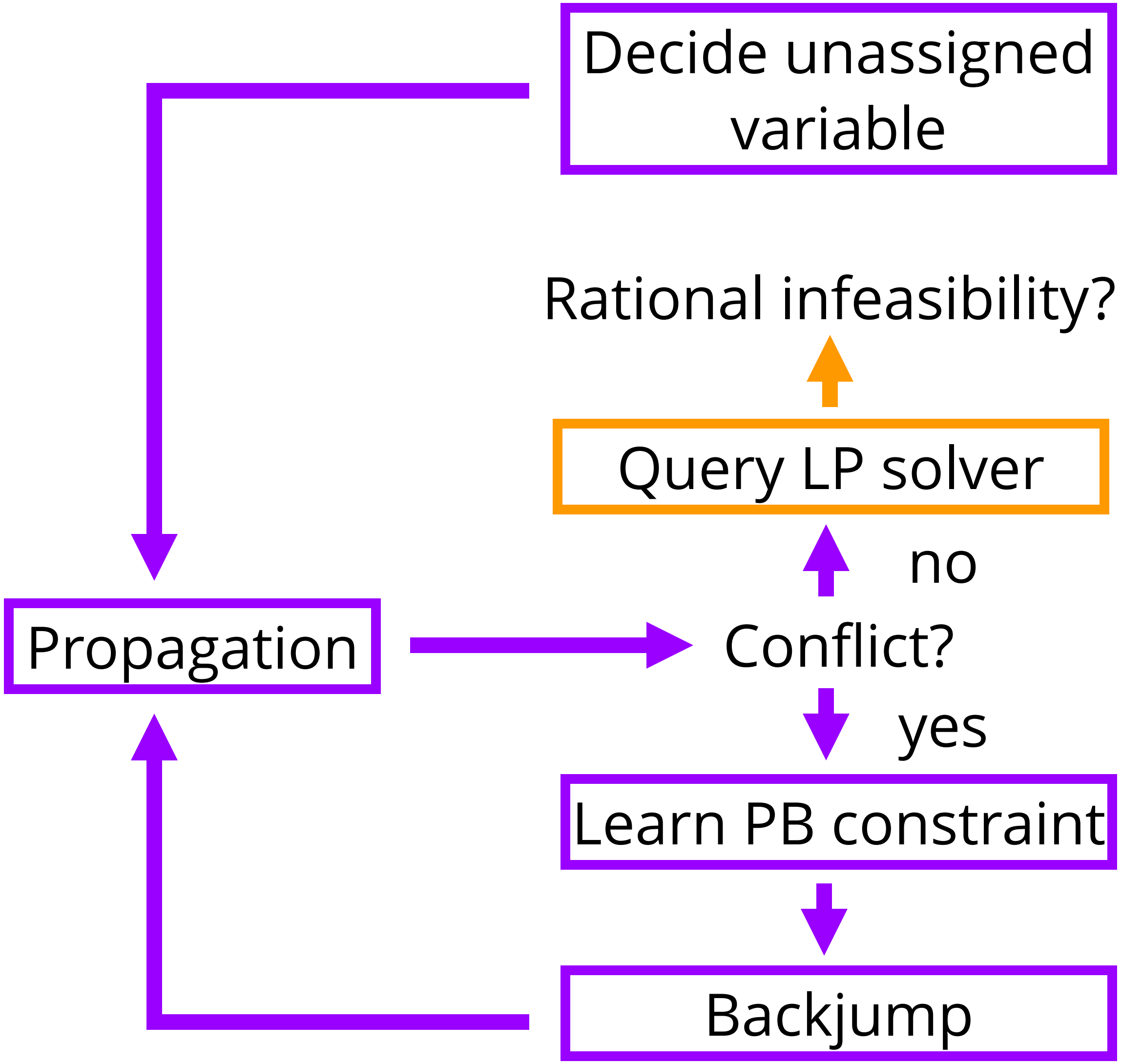


PB search loop

with LP solver call



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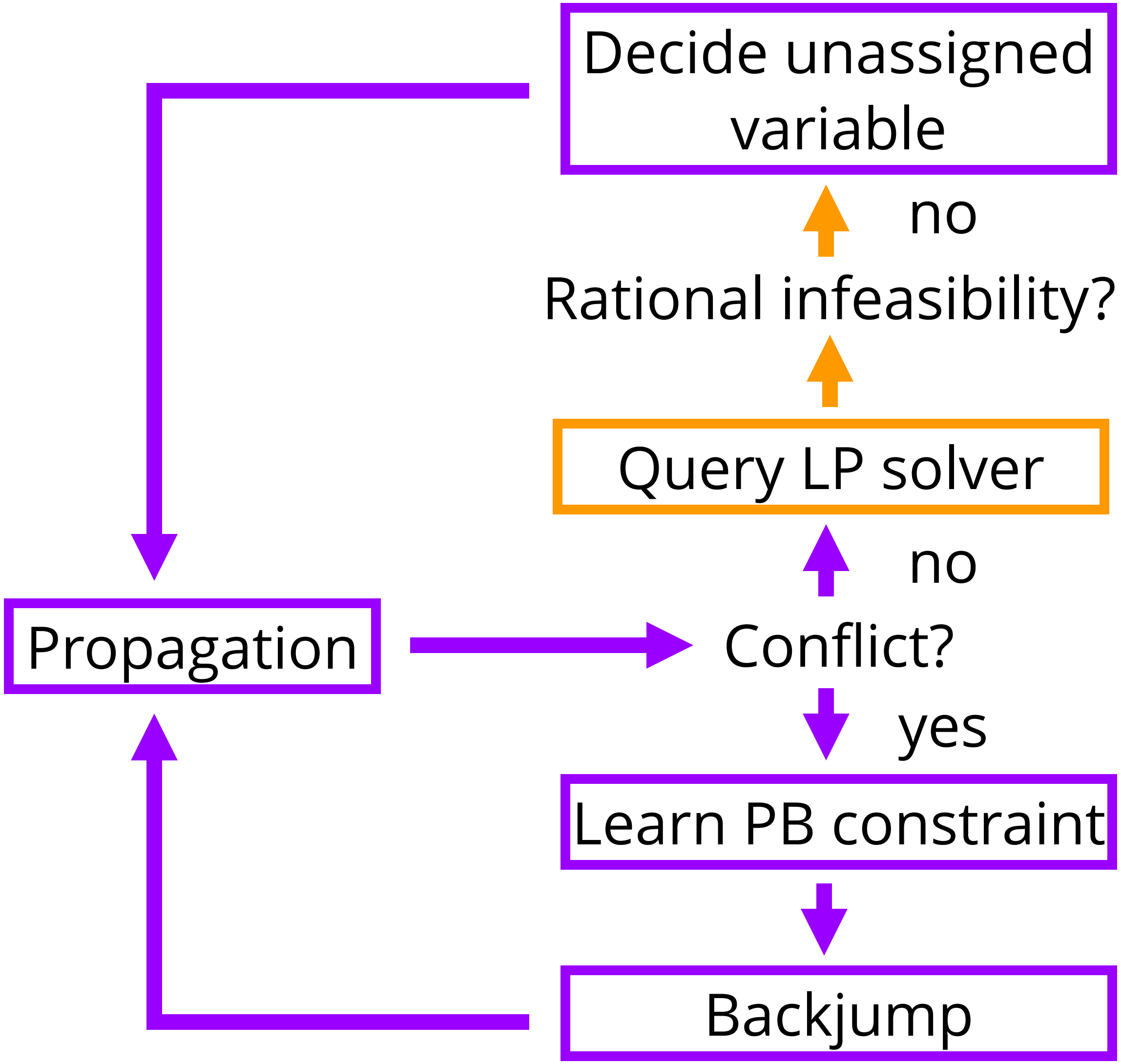

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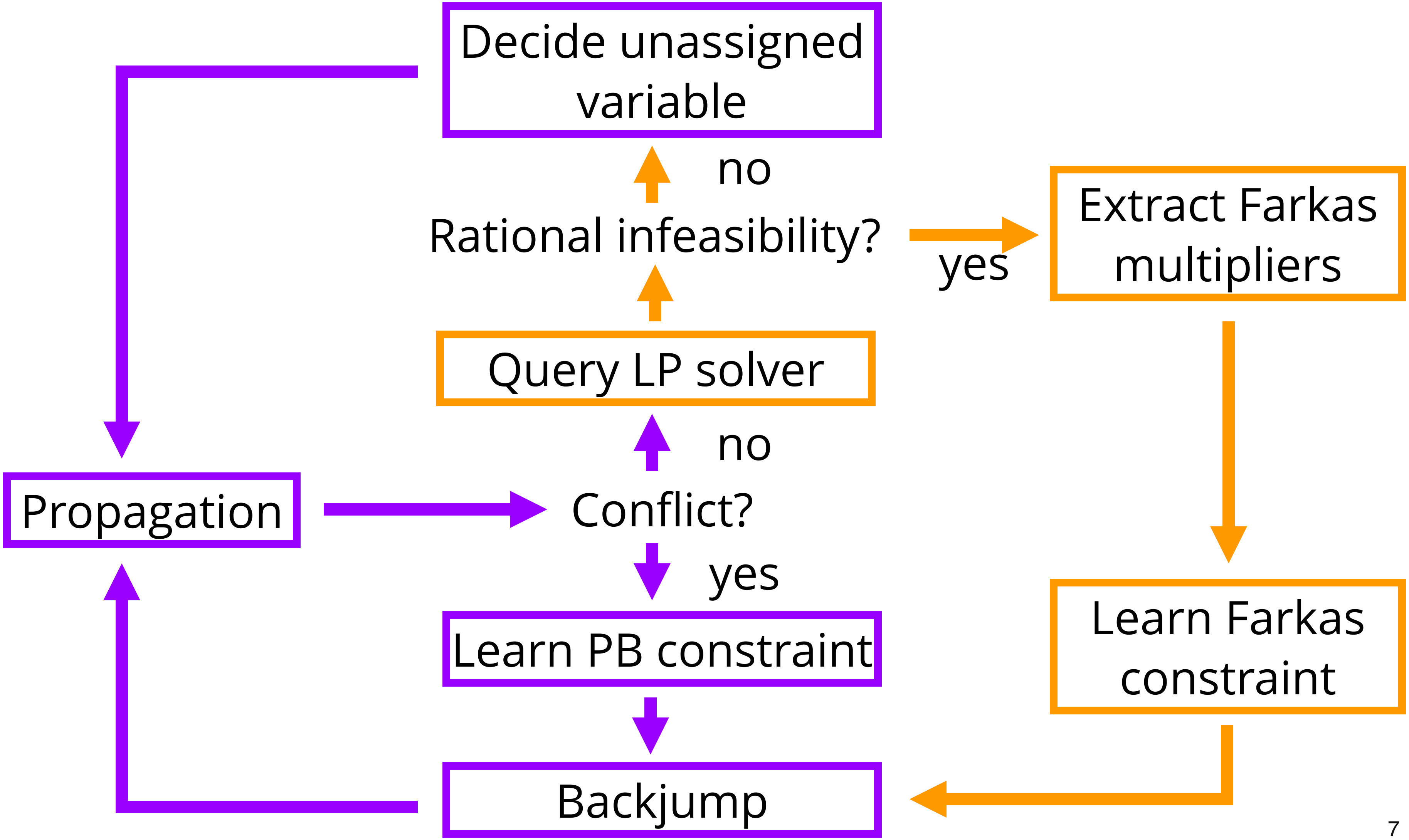

Two technical hurdles

- LP solvers are **slow** compared to PB search loop
  - Limit calls to LP solver
  - Limit LP solver running time
  - Deterministic measure: compare **#conflicts** in PB solver to **#pivots** in LP solver

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- LP solvers are **slow** compared to PB search loop
  - Limit calls to LP solver
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  - Deterministic measure: compare **#conflicts** in PB solver to **#pivots** in LP solver
- Learned constraint must be implied by input formula
  - LP solver uses inexact floating point arithmetic
  - **Recalculate** Farkas constraint with exact arithmetic
  - **Verify** Farkas constraint is still conflicting

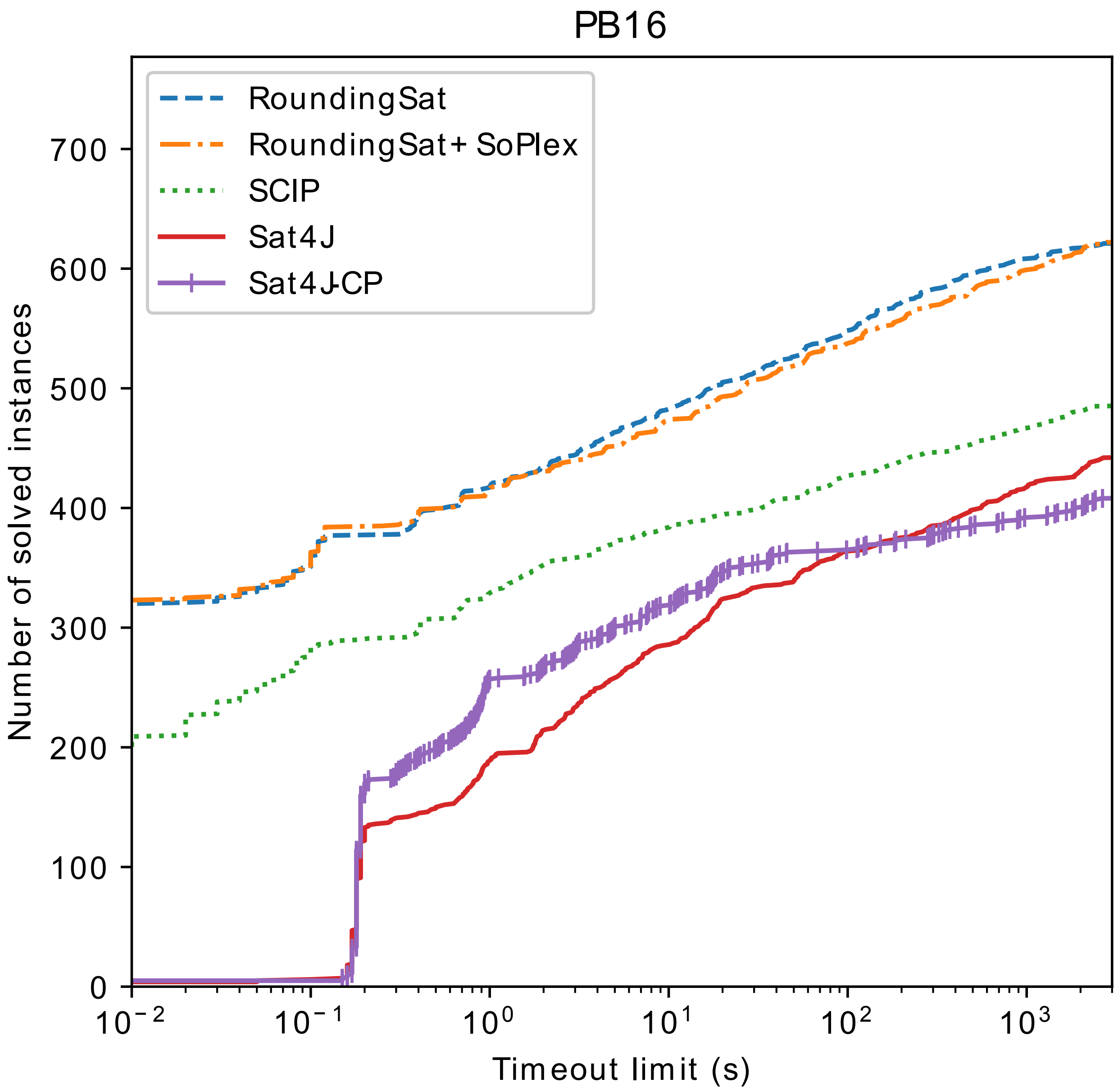
Working implementation

- PB solver **RoundingSat** [EN18]
  - Native cutting plane proofs
  - Performed well in past PB competitions
- LP solver **SoPlex** [ZIB]
  - SCIP's native LP solver
  - Fast
  - Open source

Experiments!

- 5 solver configurations
  - RoundingSat
  - **RoundingSat+SoPlex**
  - SCIP
  - Sat4J
  - Sat4J-CP
- 3000s on 16GiB machines
- 4 benchmark families:
  - PB12
  - PB16
  - MIPLIB
  - PROOF

Performance experiment



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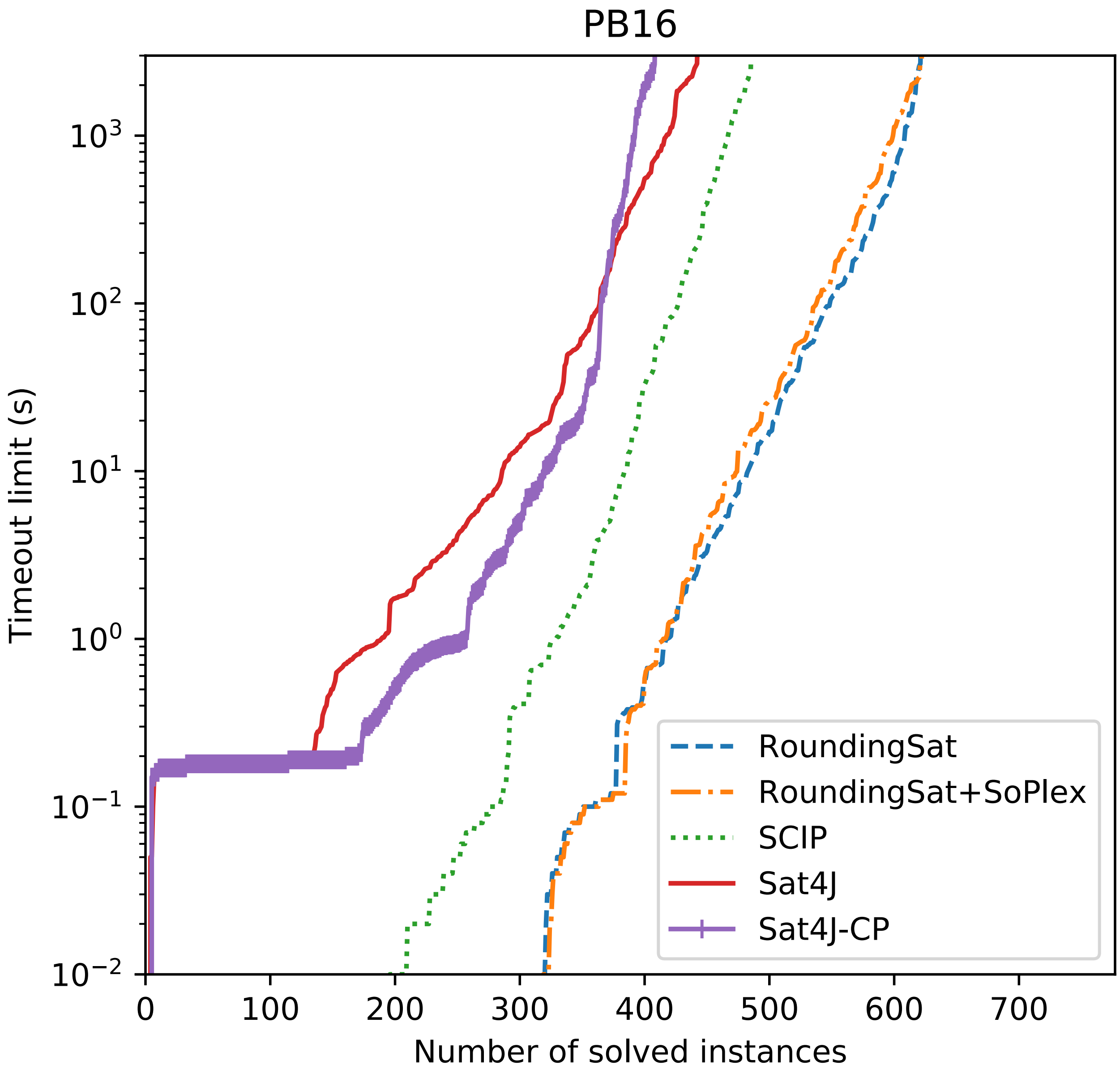

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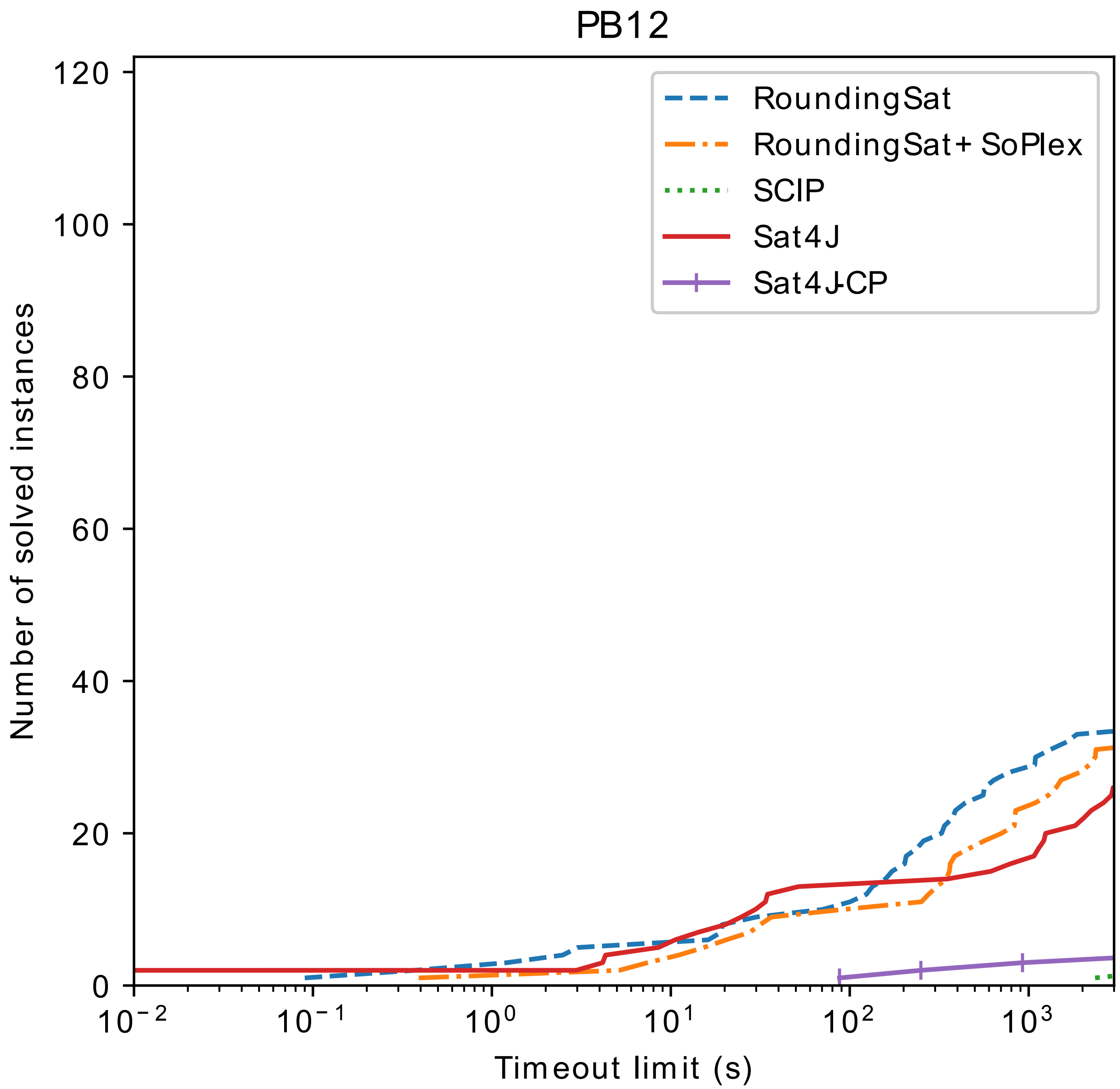

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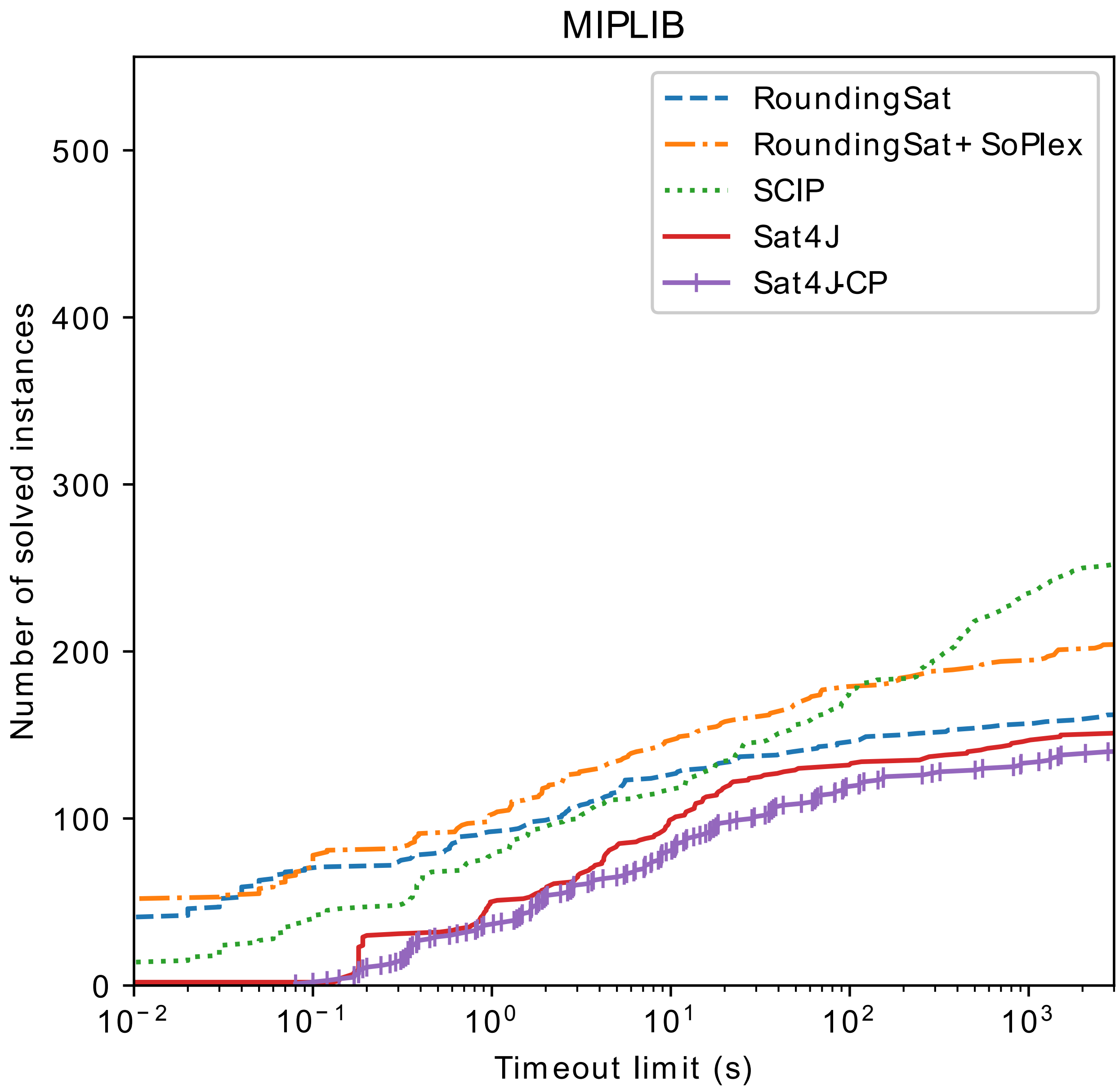

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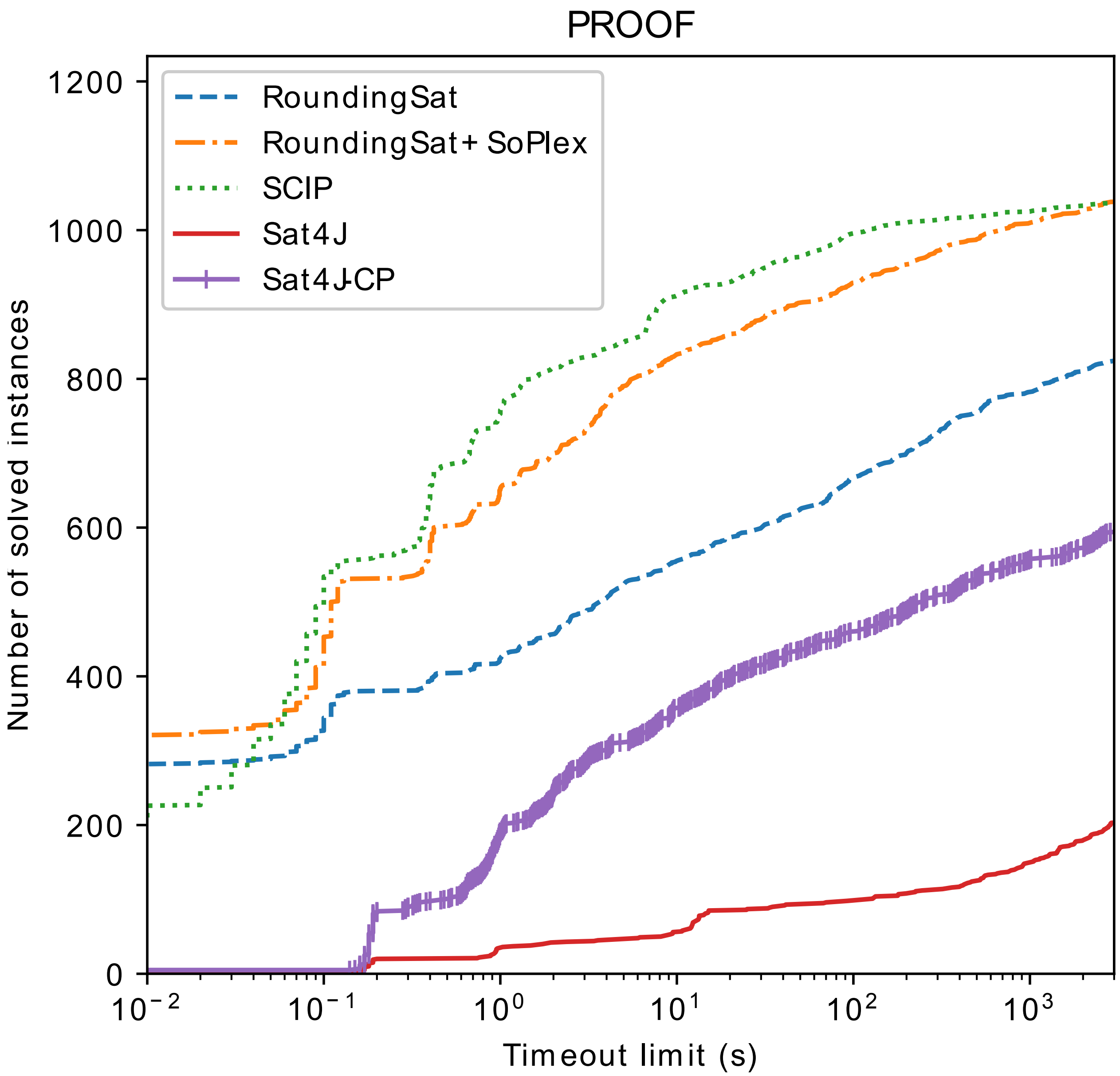

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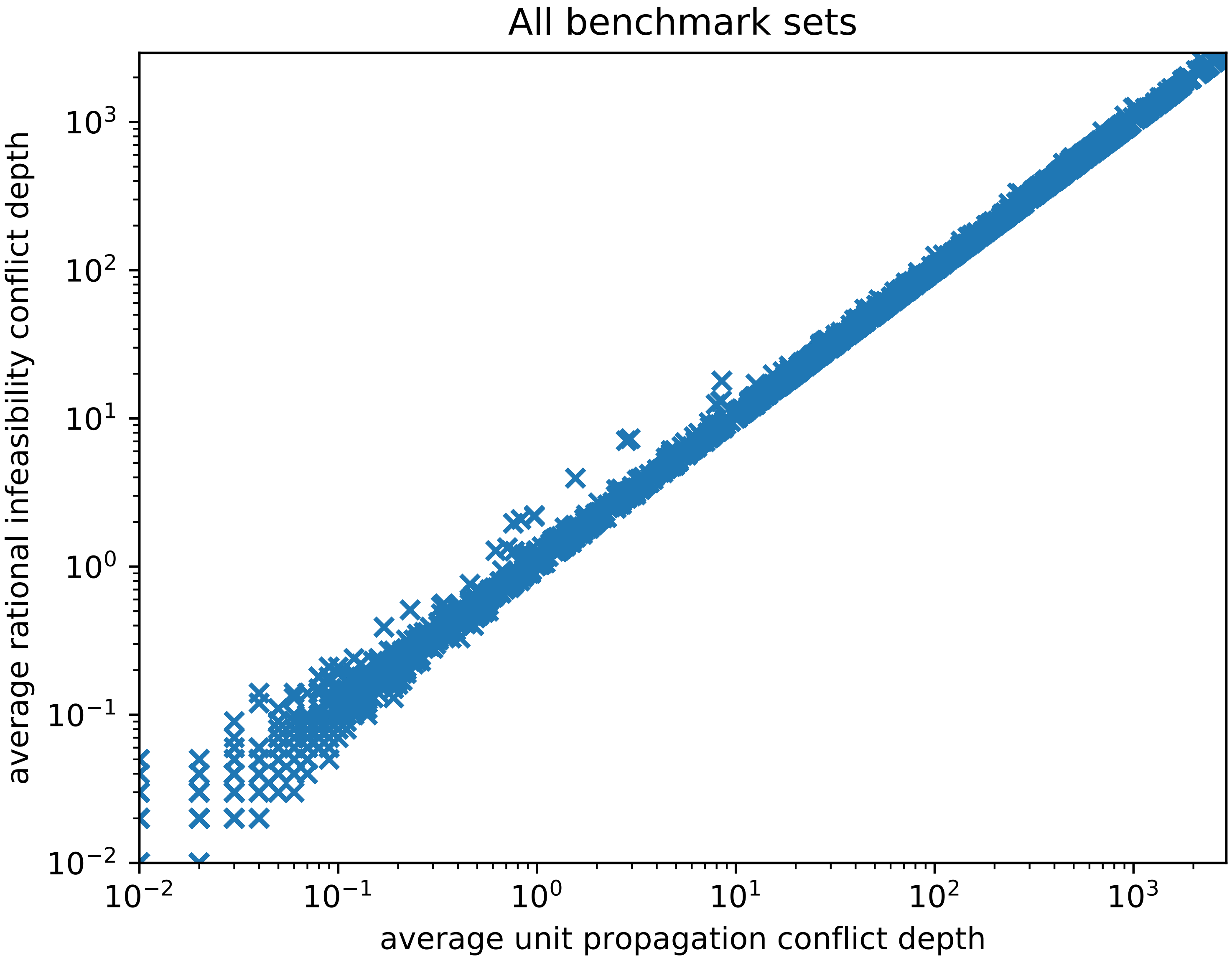
- SoPlex does not like PB12

Conflict depth experiment



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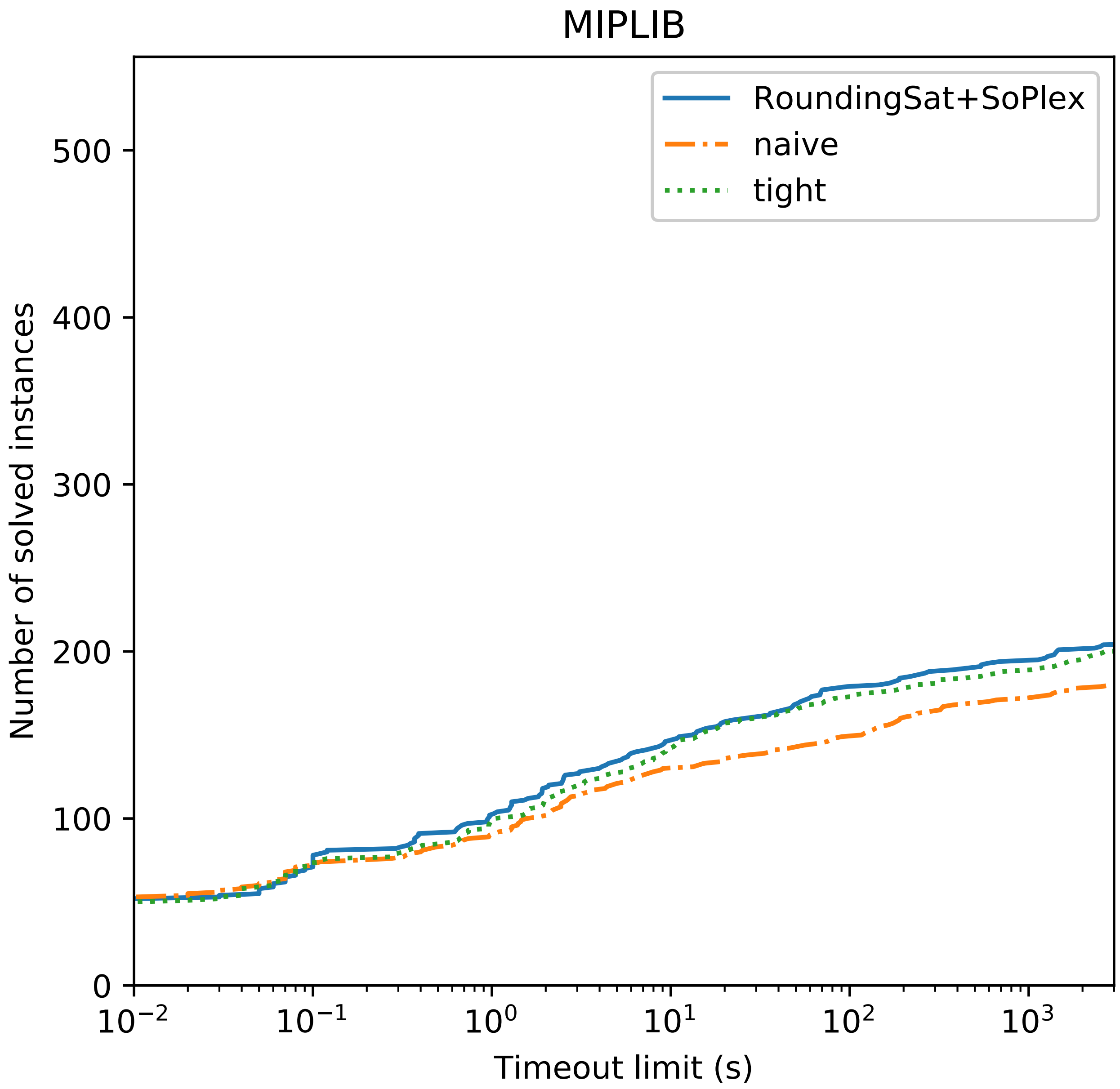
- Technique detects rational infeasibility also in **deep search nodes**

Learned constraint addition experiment



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- Other hypotheses?

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Thanks for your attention!

Questions?

References

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